arXiv:1001.0373v1 [cond-mat.str-el] 3 Jan 2010

# ORDERED STATES AND NONLINEAR LARGE-SCALE EXCITATIONS IN A PLANE MAGNET WITH SPIN $s = 1$

**J.M. BERNATSKA, P.I. HOLOD**

**National University "Kyiv-Mohyla Academy"**
*(2, Skovoroda Str., Kyiv 04070, Ukraine),*

[1]**M.M. Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine**
*(14b, Metrolohichna Str., Kyiv 03143, Ukraine)*

UDC 538.221

©2008

We study ordered states and topological excitations in a quasi-two-dimensional magnet, modeled by a square lattice with spins $s = 1$ at all sites, and the Hamiltonian with biquadratic exchange interaction between nearest neighbor sites. We propose two effective Hamiltonians for description of large-scale excitations in the two-dimensional case. They describe excitations of the mean field in a nematic phase and a mixed ferromagnetic-nematic phase. It is shown that the effective Hamiltonians are minimized on configurations with fixed topological charge. These topological excitations can arise at low temperatures and cause a destruction of a long-range order in the two-dimensional system.

## 1. Introduction

Quasi-two-dimensional magnets have various technological applications. They serve as magnetic films used for recording of information, thin ferromagnetic layers in Josephson semiconductor junctions, layered resistive systems, *etc.*

Here, we will not deal with applied aspects of theory of magnetism. However, we note that a study of two-dimensional systems has also a significant value. Studying ordered states, their stability, and excitation spectra, we obtain model scenarios of a self-organization of a substance with decrease in a temperature or under an action of external fields. To support the above-presented assertion, it is worth to recall an important role played by the Onsager's results [1] on the two-dimensional Ising model, or the Kosterlitz–Thouless theory of topological phase transitions [2,3]. This is also related to the study of the two-dimensional O(3)-sigma model or a planar Heisenberg magnet [4]. As a natural continuation of this trend, we mention a significant number of scientific papers devoted to investigations of two-dimensional continuous or lattice systems with high spins at sites.

In the present paper we consider a generalized Heisenberg magnet taking into account bilinear and biquadratic interactions at nearest-neighbor sites of a square or cubic lattice. This Hamiltonian for magnets of spin $s = 1$ was proposed and studied long ago [5, 6] without any restriction on dimensionality. At the beginning of 1970s, an existence of ordered phases different from the ferromagnetic or antiferromagnetic ones was established by the mean-field methods. In particular, if the constant of biquadratic interaction is larger than that of bilinear interaction, then a pure quadrupole ordering or a spin nematic state can be realized in the system [7, 8].

It is known that a two-dimensional system with a continuous group of symmetries has no long-range order at $T > 0$ (the Mermin–Wagner theorem). In many cases, an instability of ordered phases in two-dimensional systems implies an existence of nonlinear topological excitations caused by small fluctuations of temperature. The role of such excitations in destruction of a long-range order is proven within the model of plane rotators [2, 3] and for the two-dimensional Heisenberg ferromagnet [4].

The main result of our work is a proof of an existence of topological excitations in the model with biquadratic interaction between nearest spins $s = 1$ at sites of a square lattice. We will consider the boundary case of a nematic phase, where the constants of bilinear and biquadratic interactions are identical. It is known that, in this case, energies of both possible phases (nematic and ferromagnetic-quadrupole ones) are equal. In order to study excitations of nematic phase, we assume that $K - J = \varepsilon$, and $\varepsilon$ is a small positive value. It is obvious that topological excitations exist at these parameters, and differ slightly from those in the case $\varepsilon = 0$. If $\varepsilon > 0$, the manifold of degeneration of a ground state of the system is deformed, but the topology is not sensitive to smooth deformations. Therefore, our conclusion of existence of topological excitations at $J = K$ remains valid also in the case $K > J > 0$.

The present paper contains two parts. The first part is a survey. In the mean field approximation, we reveal existence conditions for ordered phases and solve an

equation of self-consistent relations for order parameters. Comparing with results of other researchers on this topic, we obtain conditions of occurrence of a nematic state. In the second part, averaging equations of motion over coherent states, and passing from a plane square lattice to a continuous plane, we obtain formulas for free energy of an inhomogeneous distribution of the mean field. Topological excitations are described in terms of the inhomogeneous distribution. Depending on a choice of an equilibrium state and a way of averaging, we get two formulas for the free energy: the first formula corresponds to excitations of a pure nematic state, and the second one is related to excitations in the state with nonzero magnetization and quadrupole moment. In the case of SU(3)-symmetry, we determine self-dual solutions of the problem of minimization. The obtained topological excitations give the absolute minimum for the free energy, and its value is proportional to a topological charge.

## 2. The Quantum Model of a Planar Magnet

Let us consider a plane square lattice containing atoms of spin $s$ at each site. Each atom is assigned by three spin operators $\{\hat{S}^1_n,\, \hat{S}^2_n,\, \hat{S}^3_n\}$ obeying the standard commutation relations

$$[\hat{S}^\alpha_n, \hat{S}^\beta_n] = i\varepsilon^{\alpha\beta\gamma}\hat{S}^\gamma_n\delta_{nm},$$

where the indices $\alpha$, $\beta$, and $\gamma$ run the values $\{1,\,2,\,3\}$ for each site $n$, and $\delta_{nm}$ is the Kronecker symbol.

Usually, such system is described by the Heisenberg Hamiltonian. As $s \geqslant 1$ we can include higher orders of the exchange interaction in the Hamiltonian. In particular, magnets with the biquadratic interaction were studied in the 1970s [9, 10]. The latter Hamiltonian will be considered in what follows. Let

$$\hat{\mathcal{H}} = -\sum_{n,\delta}\{J(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta}) + K(\hat{\boldsymbol{S}}_n, \hat{\boldsymbol{S}}_{n+\delta})^2\}, \qquad (1)$$

where $\hat{\boldsymbol{S}}_n$ denotes the vector $(\hat{S}^1_n,\, \hat{S}^2_n,\, \hat{S}^3_n)$ of the spin operators at site $n$, and $\delta$ runs over the nearest-neighbour sites. We assume that the exchange integrals $J$ and $K$ are positive, i.e. we consider mainly the ferromagnetic interaction.

The operators $\{\hat{S}^\alpha\}$ are defined over the $(2s+1)$-dimensional space of an irreducible representation of the group SU(2). The spin operators generate the complete matrix algebra over this space (the Burnside theorem). With respect to the adjoint action $\mathrm{ad}_{\hat{S}^\alpha}$, the complete matrix algebra is divided into a direct sum of irreducible collections of tensor operators. For example, let us consider the case of $s=1$. Then for the complete matrix algebra over the representation space, we have $\dim \mathrm{Mat}_{3\times 3} \simeq [9] = [1]+[3]+[5]$. Obviously, bases in the three- and five-dimensional irreducible collections are formed, respectively, by the operators $\hat{S}^\alpha$, and by the tensor operators of weight 2. The latter are the operators of quadrupole moment chosen in the form

$$\hat{Q}^{\alpha\beta}_n = \hat{S}^\alpha_n\hat{S}^\beta_n + \hat{S}^\beta_n\hat{S}^\alpha_n,\ \alpha \neq \beta,$$

$$\hat{Q}^{[2,2]}_n = (\hat{S}^1_n)^2 - (\hat{S}^2_n)^2,$$

$$\hat{Q}^{[2,0]}_n = \sqrt{3}\big((\hat{S}^3_n)^2 - \tfrac{2}{3}\big).$$

A normalization of the operators $\hat{S}^\alpha$ is defined by the relation

$$(\hat{S}^1)^2 + (\hat{S}^2)^2 + (\hat{S}^3)^2 = s(s+1)\mathbb{I}_3,$$

which yields $\mathrm{Tr}(\hat{S}^\alpha)^2 = \frac{1}{3}\,s\,(s+1)(2s+1)$. For $s=1$ we have $\mathrm{Tr}(\hat{S}^\alpha)^2 = 2$. We extend such normalization for all other operators.

Now we fix the canonical basis $\{|{+1}\rangle,\, |{-1}\rangle,\, |0\rangle\}$ in the representation space. Then a matrix representation of the operators of spin and quadrupole moment is as follows:

$$\hat{S}^1_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix},\ \hat{S}^2_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&-i\\0&0&i\\i&-i&0\end{pmatrix},$$

$$\hat{S}^3_n = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad \hat{Q}^{[2,0]}_n = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

$$\hat{Q}^{12}_n = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\ \hat{Q}^{13}_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&-1\\1&-1&0\end{pmatrix},$$

$$\hat{Q}^{23}_n = \frac{1}{\sqrt{3}}\begin{pmatrix}0&0&-i\\0&0&-i\\i&i&0\end{pmatrix},\ \hat{Q}^{[2,2]}_n = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

It is easy to see that the proposed matrices are connected with the Gell-Mann matrices $\hat{\Lambda}_a$, $a=1,\,2,\,\ldots,\,8$, which also form a basis in $i\mathfrak{su}(3)$. The connection is given by the linear transformations

$$\hat{S}^1_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_4 + \hat{\Lambda}_6),\ \hat{S}^2_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_5 - \hat{\Lambda}_7),\ \hat{S}^3_n = \hat{\Lambda}_3,$$

$$\hat{Q}^{12}_n = \hat{\Lambda}_2,\quad \hat{Q}^{[2,0]}_n = \hat{\Lambda}_8,\quad \hat{Q}^{[2,2]}_n = \hat{\Lambda}_1,$$

$$\hat{Q}^{13}_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_5 + \hat{\Lambda}_7),\quad \hat{Q}^{23}_n = \tfrac{1}{\sqrt{2}}(\hat{\Lambda}_4 - \hat{\Lambda}_6).$$

By $\{\hat{P}^a\}$ we denote the collection of operators $\{\hat{S}_n^1$, $\hat{S}_n^2$, $\hat{S}_n^3$, $\hat{Q}_n^{12}$, $\hat{Q}_n^{13}$, $\hat{Q}_n^{23}$, $\hat{Q}_n^{[2,2]}$, $\hat{Q}_n^{[2,0]}\}$. It is easy to prove that the commutation relations

$$[\hat{P}_n^a,\, \hat{P}_m^b] = iC_{abc}\hat{P}_n^c\delta_{nm},$$

hold true. Here, the tensor of structure constants $C_{abc}$ is antisymmetric under a permutation of any pair of indices, and its nonzero components are

$$C_{123} = C_{145} = C_{167} = C_{264} = C_{257} = C_{356} = 1,$$

$$C_{168} = C_{528} = \sqrt{3}, \quad C_{437} = 2,$$

In terms of the operators of spin and quadrupole moment, Hamiltonian (1) gets the form

$$\hat{\mathcal{H}} = -\left(J - \tfrac{1}{2}K\right)\sum_{n,\delta}\sum_{\alpha}\hat{S}_n^\alpha, \hat{S}_{n+\delta}^\alpha -$$

$$-\tfrac{1}{2}K\sum_{n,\delta}\sum_{a}\hat{Q}_n^a\hat{Q}_{n+\delta}^a - \tfrac{4}{3}KN, \tag{2}$$

where $N$ denotes the total number of sites of the lattice. Obviously Hamiltonian remains SU(2)-invariant; hence, the operators $\hat{S}_n^\alpha$ and $\hat{Q}_n^a$ are transformed by formulas of the adjoint representation

$$\hat{U}\hat{S}_n^\alpha\hat{U}^{-1} = \sum_\beta \hat{D}^{\alpha\beta}(\hat{U})\hat{S}_n^\beta,$$

$$\hat{U}\hat{Q}_n^a\hat{U}^{-1} = \sum_b \hat{D}^{ab}(\hat{U})\hat{Q}_n^b, \quad \forall\hat{U} \in \mathrm{SU}(2),$$

where $\hat{D}^{\alpha\beta}(\hat{U})$ and $\hat{D}^{ab}(\hat{U})$ are matrices of the real irreducible representations of SU(2) with dimensions 3 and 5, respectively. If $K = J$, then the SU(2)-symmetry can be extended to the group SU(3). In this case, the Hamiltonian (2) gets the form

$$\hat{\mathcal{H}} = -\tfrac{1}{2}J\sum_{n,\delta}\sum_a \hat{P}_n^a\hat{P}_{n+\delta}^a - \tfrac{4}{3}JN. \tag{3}$$

To study possible ordered phases of such spin system, we use the mean-field approximation.

## 3. Mean-Field Approximation. Ordered States

Now we replace the interaction between spin and quadrupole operators that is described by Hamiltonian (2) with an effective interaction between the operators $\hat{P}_n^a$ and the classical mean field. Components of the mean field are considered proportional to averages (quasiaverages) of the quantum operators $\{\hat{P}_n^a\}$. The Hamiltonian in the mean field approximation has the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)\sum_{n,\delta}\sum_\alpha \hat{S}_n^\alpha\langle\hat{S}_{n+\delta}^\alpha\rangle -$$

$$-\tfrac{1}{2}K\sum_{n,\delta}\sum_a \hat{Q}_n^a\langle\hat{Q}_{n+\delta}^a\rangle - \tfrac{4}{3}KN. \tag{4}$$

It is worth to give a warning that a direct calculation of the averages $\{\langle\hat{S}_n^\alpha\rangle\}$ and $\{\langle\hat{Q}_n^a\rangle\}$, for example by means of the density matrix $\hat{\rho}(T) = \exp\{-\frac{\mathcal{H}}{kT}\}$, results in the zero values. This follows from the SU(2)-symmetry of Hamiltonian (2). Nonzero values of the averages appear if the symmetry is broken. Symmetry breaking can be stimulated by an external field, which vanishes after specifying an order in the magnetic system. The quantities calculated in this way are called "quasiaverages" [11].

Hence, we assume that in our system the nonzero quasiaverages $\{\langle\hat{S}_n^\alpha\rangle\}$ and $\{\langle\hat{Q}_n^a\rangle\}$ exist, and form a classical 8-component vector field $\mu_a(x_n)$, $a = 1, 2, \ldots, 8$. In order to obtain nonzero values of $\{\langle\hat{Q}_n^a\rangle\}$, the external field must have nonzero gradient. If the mean field is homogeneous, an action of the group SU(2) transforms Hamiltonian (4) to the form

$$\hat{\mathcal{H}}_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)\sum_n \hat{S}_n^3\langle\hat{S}^3\rangle -$$

$$-\tfrac{1}{2}K\sum_n \hat{Q}_n^{[2,0]}\langle\hat{Q}^{[2,0]}\rangle - \tfrac{4}{3}KN = -\tfrac{4}{3}KN -$$

$$-\sum_n\left\{(J - \tfrac{1}{2}K)\hat{S}_n^3\mu_3 + \tfrac{1}{2}K\hat{Q}_n^{[2,0]}\mu_8\right\}.$$

In the case of thermodynamic equilibrium and an infinite lattice, the fields $\mu_3 = \langle\hat{S}^3\rangle$ and $\mu_8 = \langle\hat{Q}^{[2,0]}\rangle$ are constant, i.e. have the same values at all points $x_n$ (a homogeneous mean field). These quantities serve as *order parameters*. Obviously, $\mu_3$ describes a normalized magnetization (a ratio of $z$-projection of magnetic moment to a saturation magnetization), and $\mu_8$ is analogously related to a quadrupole moment.

In the mean field approximation, it is easy to calculate a partition function for the homogeneous case

$$NZ(\mu_3, \mu_8, T) = \operatorname{Tr} e^{-\frac{\mathcal{H}_{\mathrm{MF}}}{kT}} = \operatorname{Tr} e^{-\frac{Nh_{\mathrm{MF}}}{kT}},$$

where $h_{\mathrm{MF}}$ denotes a one-site Hamiltonian

$$h_{\mathrm{MF}} = -(J - \tfrac{1}{2}K)\mu_3\hat{S}^3 - \tfrac{1}{2}K\mu_8\hat{Q}^{[2,0]} - \tfrac{4}{3}K. \tag{5}$$

The introduced mean field makes sense if *self-consistent relations* are held:

$$\mu_3 = \langle\hat{S}^3\rangle_{MF} = \frac{\operatorname{Tr}\hat{S}^3 e^{-\frac{Nh_{\mathrm{MF}}}{kT}}}{\operatorname{Tr} e^{-\frac{Nh_{\mathrm{MF}}}{kT}}},$$

$$\mu_8 = \langle \hat{Q}^{[2,0]} \rangle_{MF} = \frac{\mathrm{Tr}\, \hat{Q}^{[2,0]} e^{-\frac{N h_{\mathrm{MF}}}{kT}}}{\mathrm{Tr}\, e^{-\frac{N h_{\mathrm{MF}}}{kT}}}.$$

these relations serve as an analog of the Weiss equation from theory of ferromagnetism. The averages of operators can be presented via the partition function:

$$\mu_3 = \frac{kT}{(J - \frac{K}{2})} \frac{\partial}{\partial \mu_3} \ln Z(\mu_3, \mu_8, T),$$

$$\mu_8 = \frac{2kT}{K} \frac{\partial}{\partial \mu_8} \ln Z(\mu_3, \mu_8, T).$$

For the system described by one-site Hamiltonian (5), the self-consistent relations get the form

$$\mu_3 = \frac{2\,\mathrm{sh}\, \frac{(J-\frac{K}{2})\mu_3}{kT}}{\exp\left\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\right\} + 2\,\mathrm{ch}\, \frac{(J-\frac{K}{2})\mu_3}{kT}},$$

$$\mu_8 = \frac{2}{\sqrt{3}} \frac{\mathrm{ch}\, \frac{(J-\frac{K}{2})\mu_3}{kT} - \exp\left\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\right\}}{\exp\left\{-\frac{\sqrt{3}\,K\mu_8}{2kT}\right\} + 2\,\mathrm{ch}\, \frac{(J-\frac{K}{2})\mu_3}{kT}}. \qquad (6)$$

Note, that the true averages are always less than their expectation values calculated from the self-consistent relations. Therefore, solutions of (6) have a qualitative sense only.

Here we analyze Eqs. (6) and make comparison with results described in the literature. The obvious solution corresponds to the paramagnetic state with $\mu_3 = 0$ and $\mu_8 = 0$; this state is realized at temperatures $kT > J - K/2$. At the same time, this inequality shows that the model, oriented to ferromagnetic materials, gives a meaningful result only in the region $J - K/2 < 0$, which contains areas with the ferromagnetic and quadrupole orderings, according to the well-known phase diagram (Fig. 1) for the bilinear-biquadratic $s = 1$ 1-dimensional spin model [12].

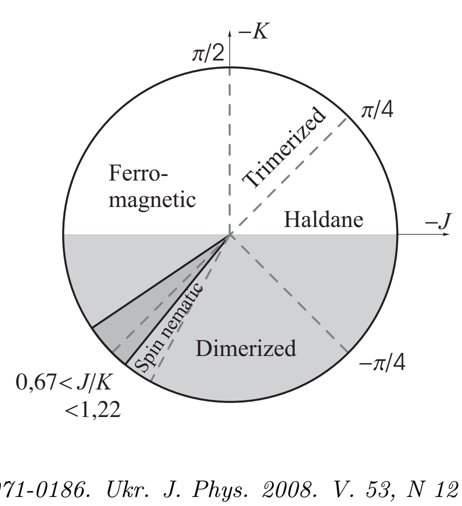


Fig. 1. Phase diagram of a one-dimensional system of spins $s = 1$

In the case of $K < 0$, the self-consistent relations have a unique nontrivial solution, corresponding to the ferromagnetic ordering, because this solution tends to the boundary values $\mu_3 = 1$ and $\mu_8 = \frac{1}{\sqrt{3}}$ as temperature decreases to zero. The critical temperature of transition from a ferromagnetic state into a paramagnetic one is determined in terms of the constants $J$ and $K$ as $T_{\mathrm{c}} = \frac{2}{3k}(J - K/2)$.

In the case of $K > 0$ (the light-grey region in Fig. 1), Eqs. (6) have more than one nontrivial solution: two solutions corresponding to ferromagnetic states with the boundary values $\mu_3^{(1)} = 1$ and $\mu_3^{(2)} = 2/3$ (and the corresponding values of $\mu_8$), and two solutions describing nematic states ($\mu_3 = 0$) with the boundary values $\mu_8^{(1)} = \frac{-2}{\sqrt{3}}$ and $\mu_8^{(2)} = \frac{1}{\sqrt{3}}$. Existence of four ordered states in ferromagnets is also reported in [7]: they are a ferromagnetic state with $\mu_3^{(1)} = s$, a quadrupole (or nematic) state with $\mu_3 = 0$, $\mu_8^{(1)} = -s(s+1)/\sqrt{3}$, a partially ordered quadrupole state with $\mu_3 = 0$, $\mu_8^{(1)} > 0$, and a partially ordered ferromagnetic state with $\mu_3^{(1)} < s$. Partially ordered states are unstable [7].

Analyzing the temperature evolution of solutions of (6) as $K > 0$, $J > 0$, we reveal two critical temperatures, which are solutions of the equation

$$2\Big(\frac{J - K/2}{kT} - 1\Big) = \exp\Big\{\frac{K}{kT}\Big(1 - \frac{3kT}{2(J - K/2)}\Big)\Big\}.$$

An obvious solution is $T_{\mathrm{c1}} = \frac{2}{3k}(J - K/2)$. The other solution $T_{\mathrm{c2}}$ is calculated numerically. In the region $J > K$, i.e. for ferromagnetic materials, the temperature $T_{\mathrm{c2}}$ is less than $T_{\mathrm{c1}}$, whereas the reversed situation takes place for nematics in the region $J < K$. At smaller critical temperature the solution $\mu_3^{(2)}$ disappears. Then only the solution $\mu_3^{(1)}$ exists in a certain interval of temperatures. A comparison with results of the paper [8] shows that at a higher critical temperature we have a second order phase transition from a ferromagnetic state into a paramagnetic one.

A somewhat different behavior is observed in materials with $J \approx K$, or more precisely $0.67 < J/K < 1.22$, i.e. on the boundary between the ferromagnetic and nematic regions (the dark-grey region in Fig. 1). Disappearing at a lower critical temperature, the solution $\mu_3^{(2)}$ arises again at a higher critical temperature, and grows continuously from zero toward $\mu_3^{(1)}$. When two solutions coincide at a certain

temperature $T_0$, they disappear by jump. We may conclude that the phase transition from a ferromagnetic state to a paramagnetic one is a transition of the first order. This well agrees with results of the paper [8], where the change of a second order phase transition into a first order one in the region $2J/3 < K < J$ was considered (for ferromagnetic materials).

Further we consider the case $J = K$, corresponding to the boundary between the ferromagnetic and quadrupole regions. Moreover, as $J = K$ the quantum Hamiltonian (2) and the Hamiltonian in the mean-field approximation are SU(3)-invariant. The latter has the form

$$\hat{\mathcal{H}}_{\text{MF}} = -\tfrac{1}{2}J\sum_n\sum_a \hat{P}^a_n\langle\hat{P}^a\rangle - \tfrac{4}{3}JN =$$

$$= -\tfrac{1}{2}J\sum_n\sum_a \hat{P}^a_n\mu_a - \tfrac{4}{3}JN. \qquad (7)$$

## 4. Equations of Motion for Large-Scale Fluctuations of the Mean Field

Now we return to the quantum SU(3)-invariant spin model with Hamiltonian (3). The Heisenberg evolution equations for the operators $\hat{P}^a_n$ have the form

$$i\hbar\frac{d\hat{P}^a_n}{dt} = [\hat{P}^a_n, \hat{\mathcal{H}}]. \qquad (8)$$

We assuming that the system is ordered state, and take an average of the both sides of Eq. (8) over Heisenberg (time independent) coherent states

$$|\psi(n)\rangle = \frac{1}{\sqrt{N}}\big(c_1(n)|1\rangle + c_{-1}(n)|-1\rangle + c_0(n)|0\rangle\big),$$

$$|c_1|^2 + |c_{-1}|^2 + |c_0|^2 = 1.$$

On the other hand, an averaging can be performed by means of the density matrix (thermodynamical averaging) as $T > 0$ [13]. In both cases, we suppose

$$\langle\hat{P}^a_n\hat{P}^b_m\rangle \approx \langle\hat{P}^a_n\rangle\langle\hat{P}^b_m\rangle = \mu_a(n)\mu_b(m), \qquad (9)$$

i.e. we neglect correlations between fluctuations of the quantum fields $\hat{P}^a_n$. Then we obtain the following system of *Hamiltonian equations* for the averages $\mu_a(n)$:

$$\hbar\frac{d\mu_a(n)}{dt} = C_{abc}\mu_b(n)\frac{\partial\langle\mathcal{H}\rangle}{\partial\mu_c(n)} = \{\mu_a(n),\,\langle\mathcal{H}\rangle\}. \quad (10)$$

Taking (9) into account we have $\langle\mathcal{H}\rangle = \langle\mathcal{H}_{\text{MF}}\rangle$.

In order to investigate large-scale fluctuations of the field $\mu_a(n)$ we take a two-dimensional continuum instead of the discrete lattice. Such transition is well known for an SU(2)-magnet [14] and underlies the macroscopic phenomenological theory of magnetism [15]. In the continuous theory dynamical variables are densities of averaged spin and quadrupole moments:

$$M_a(\boldsymbol{x}) = \lim_{S\to 0}\frac{1}{S}\sum_{n\in S}\mu_a(n)\delta_{\boldsymbol{x},\boldsymbol{x}_n} = \sum_{n\in S}\mu_a(n)\delta(\boldsymbol{x}-\boldsymbol{x}_n).$$

Here, $S$ is a 'physically' infinitesimal region of the lattice, and $\delta(\boldsymbol{x}-\boldsymbol{x}_n)$ is the Dirac delta-function, which has the dimension of reciprocal area. A Poisson bracket on the space of $\{M_a(\boldsymbol{x})\}$ is calculated by the formula

$$\{M_a(\boldsymbol{x}), M_b(\boldsymbol{y})\} = C_{abc}M_c(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}).$$

In what follows, we deal with the dimensionless field $\mu_a(\boldsymbol{x}) = l^2 M_a(\boldsymbol{x})$, where $l$ is a distance between the nearest neighbors of the square lattice.

Considering $(j\pm 1, k)$, $(j, k\pm 1)$ as the nearest neighbors of the site $n = (j,k)$, in Eqs. (10) we perform a transition from the discrete variable $\boldsymbol{x}_n$ to a continuous one $\boldsymbol{x}$. Then the field $\{\mu_a(\boldsymbol{x})\}$ satisfies the equations

$$\hbar\frac{\partial\mu^a(\boldsymbol{x})}{\partial t} = \{\mu_a(\boldsymbol{x}),\,\mathcal{H}_{\text{eff}}\} = -C_{abc}\mu_b\frac{\delta\mathcal{H}_{\text{eff}}}{\delta\mu_c}, \quad (11)$$

where

$$\mathcal{H}_{\text{eff}} = J\int\sum_a\Big(\frac{\partial\mu_a}{\partial\boldsymbol{x}}\Big)^2 d^2\boldsymbol{x}.$$

A suitable representation of the system of Hamilton equations (11) is the matrix equation

$$\frac{\partial\hat{\mu}}{\partial t} = \frac{2Jl^2}{\hbar}[\hat{\mu}, \Delta\hat{\mu}], \qquad (12)$$

where

$$\hat{\mu} = -\frac{i}{2}\sum_a\mu_a(\boldsymbol{x})\hat{P}^a.$$

Obviously, $\hat{\mu}$ is a Hermitian $3\times 3$ matrix, the bracket $[\cdot,\cdot]$ denotes a matrix commutator, and $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplace operator over a two-dimensional space. Equation (11) generalizes the Landau–Lifshits equation [16] for isotropic ferromagnets to the case of 8-component mean field $\mu_a(\boldsymbol{x})$. The Landau–Lifshits equation is well-known in the macroscopic theory of magnetism, and suitable for describing large-scale excitations in planar magnets and exploring the ferromagnetic resonance.

## 5. Free Energy and Topological Charge

As mentioned above, a Hamiltonian coincides with a free energy at constant temperature and volume. Therefore, we use the notion of free energy in what follows. As shown in Appendix (Section 6) by means of an algebraic approach, Eq. (12) coincides with the two-dimensional generalization of Eq. (25) on a degenerate orbit. This equation is a Hamiltonian one, though it is nonintegrable in the two-dimensional case; and its Hamiltonian can be used as an effective free energy for the spin system in question, namely:

$$\mathcal{F}_1^{\mathrm{eff}} = \frac{2}{3h_0}\iint \left(\mu_{a,x}^2 + \mu_{a,y}^2\right) dxdy.$$

Obviously, $\mathcal{F}_1^{\mathrm{eff}}$ is a part of the total free energy of a magnet, and arises from an inhomogeneous distribution of the average values $\{\mu_a(x)\}$.

The algebraic approach yields one more equation of motion, associated with a generic orbit. Evidently, this equation can be obtained from the quantum-mechanical approach, by performing a relevant averaging that takes correlations into account. Therefore, we consider another effective free energy

$$\mathcal{F}_2^{\mathrm{eff}} = \frac{1}{2(3f_0^2 - h_0^3)}\iint \bigl(h_0^2(\mu_{a,x}^2 + \mu_{a,y}^2) -$$

$$-6f_0(\mu_{a,x}T_{a,x} + \mu_{a,y}T_{a,y}) + 3h_0(T_{a,x}^2 + T_{a,y}^2)\bigr)\, dxdy.$$

Equations of extremals for the functionals of free energy are the two-dimensional generalization of Eqs. (25) and (26).

If equations of constrains determining orbits are solved or, in other words, orbits are parameterized, then the formulas for the free energy can be simplified. Orbits of co-adjoint representation of semisimple compact Lie groups are compact Kählerian manifolds. Therefore, it is suitable to use a complex parameterization. In order to parameterize a generic orbit, it requires three complex variables $w_1$, $w_2$, d $w_3$. Explicit formulas for the parameterization of a generic orbit are the following:

$$\mu_1 = \frac{m - \sqrt{3}\,q}{2\sqrt{2}} \cdot \frac{w_2 + w_3 + \bar{w}_2 + \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} - \frac{m}{\sqrt{2}}\frac{(1 - w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1 - \bar{w}_1)(w_3 - w_1 w_2)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_2 = \frac{m - \sqrt{3}\,q}{2i\sqrt{2}} \cdot \frac{w_3 - w_2 - \bar{w}_3 + \bar{w}_2}{1 + |w_2|^2 + |w_3|^2} - \frac{im}{\sqrt{2}}\frac{(1 + w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) - (1 + \bar{w}_1)(w_3 - w_1 w_2)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_3 = -\frac{m - \sqrt{3}\,q}{2} \cdot \frac{|w_2|^2 - |w_3|^2}{1 + |w_2|^2 + |w_3|^2} + \frac{m(1 - |w_1|^2)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_4 = \frac{m - \sqrt{3}\,q}{2i} \cdot \frac{\bar{w}_2 w_3 - w_2\bar{w}_3}{1 + |w_2|^2 + |w_3|^2} + \frac{im(w_1 - \bar{w}_1)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_5 = \frac{m - \sqrt{3}\,q}{2\sqrt{2}} \cdot \frac{w_3 - w_2 + \bar{w}_3 - \bar{w}_2}{1 + |w_2|^2 + |w_3|^2} - \frac{m}{\sqrt{2}}\frac{(1 + w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2) + (1 + \bar{w}_1)(w_3 - w_1 w_2)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_6 = \frac{m - \sqrt{3}\,q}{2i\sqrt{2}} \cdot \frac{w_2 + w_3 - \bar{w}_2 - \bar{w}_3}{1 + |w_2|^2 + |w_3|^2} + \frac{im}{\sqrt{2}}\frac{(1 - \bar{w}_1)(w_3 - w_1 w_2) - (1 - w_1)(\bar{w}_3 - \bar{w}_1\bar{w}_2)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_7 = \frac{m - \sqrt{3}\,q}{2} \cdot \frac{\bar{w}_2 w_3 + w_2\bar{w}_3}{1 + |w_2|^2 + |w_3|^2} - \frac{m(w_1 + \bar{w}_1)}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2},$$

$$\mu_8 = -\frac{m - \sqrt{3}\,q}{2\sqrt{3}} \cdot \frac{2 - |w_2|^2 - |w_3|^2}{1 + |w_2|^2 + |w_3|^2} + \frac{m}{\sqrt{3}} \cdot \frac{1 + |w_1|^2 - 2|w_3 - w_1 w_2|^2}{1 + |w_1|^2 + |w_3 - w_1 w_2|^2}. \tag{13}$$

Here, $m$ and $q$ denote boundary values of the quantities $\mu_3$ and $\mu_8$, respectively. For a degenerate orbit, it is sufficient to have two complex variables, for instance $w_2$ and $w_3$; in this case, we assign $w_1 = 0$.

After the restriction onto an orbit by formulas (13), the expressions for free energy get the form

$$\mathcal{F}^{\mathrm{eff}} = \int \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} \left(\frac{\partial w_\alpha}{\partial z}\frac{\partial \bar{w}_\beta}{\partial \bar{z}} + \frac{\partial w_\alpha}{\partial \bar{z}}\frac{\partial \bar{w}_\beta}{\partial z}\right) dz d\bar{z}, \tag{14}$$

where $g_{\alpha\bar{\beta}}$ denote components of a metrics on the orbit,

and real coordinates $x$, $y$ on a plane are changed into complex ones $z$, $\bar{z}$.

While coadjoint orbits are Kählerial manifolds they possess Kähler potentials, which generate related metrics tensor $g$ and 2-form $h$; their components are calculated by the formulas

$$g_{\alpha\bar{\beta}} = \frac{\partial^2\Phi}{\partial w_\alpha \partial\bar{w}_\beta}, \qquad h_{\alpha\bar{\beta}} = i\frac{\partial^2\Phi}{\partial w_\alpha \partial\bar{w}_\beta}.$$

A 2-form gives rise to a topological charge

$$Q = \frac{1}{4\pi}\int \sum_{\alpha\beta} h_{\alpha\bar{\beta}} \left(\frac{\partial w_\alpha}{\partial z}\frac{\partial\bar{w}_\beta}{\partial\bar{z}} - \frac{\partial w_\alpha}{\partial\bar{z}}\frac{\partial\bar{w}_\beta}{\partial z}\right) dz\wedge d\bar{z},$$

which means a degree of mapping of a plane into an orbit, realized by the function $w_1$, $w_2$, $w_3$.

On a degenerate orbit of SU(3) the function $\Phi_2 = \ln(1+|w_2|^2+|w_3|^2)$ serves as a Kählerian potential, and the metrics tensor from (14) is a Kählerian one. Then the formulas for topological charge and free energy differ only in a sign ('+' for a free energy, and '−' for a topological charge), hence,

$$\mathcal{F}[\xi] \geqslant 4\pi|Q|.$$

The equality holds if the second term in the brackets vanishes, that happens if the functions $\{w_\alpha\}$ are holomorphic or antiholomorphic.

Consequently, holomorphic functions form a class of solutions with quadrupole ordering ($m=0$) that correspond to the minimum of free energy. The same takes place for antiholomorphic functions.

Now we consider the case of a generic orbit. The cohomology class of rank 2 for the orbit is two-dimensional, then there exist two basis 2-forms, generated by the Kählerian potentials $\Phi_2$, and $\Phi_1 = \ln(1+|w_1|^2+|w_3-w_1w_2|^2)$. As a unique Kählerian potential we take the one corresponding to the Kirillov-Costant-Suriau form

$$\Phi = m\Phi_1 - \frac{m-\sqrt{3}\,q}{2}\,\Phi_2. \tag{15}$$

Generally speaking, the metrics tensor of free energy (14) is not Kählerian. However, we can construct a Kählerian metrics of the form

$$\mathcal{F}_2^{\mathrm{eff}} = \frac{1}{2(3f_0^2-h_0^3)}\iint\Big(C_1(\mu_{a,x}^2+\mu_{a,y}^2)-$$

$$+C_2(\mu_{a,x}T_{a,x}+\mu_{a,y}T_{a,y}) + C_3(T_{a,x}^2+T_{a,y}^2)\Big)\,dxdy,$$

by choosing appropriate coefficients $C_1$, $C_2$, $C_3$.

Then all conclusions relative to a degenerate orbit can be extended to a generic one. That is, the class of solutions with ferromagnetic ordering that correspond to the minimum of free energy are realized by holomorphic (or antiholomorphic) functions.

### *5.1. Large-scale topological excitations*

Excitations of a state with quadrupole ordering are described by spatially inhomogeneous distributions of the 8-component vector field $\mu_a(x)$, living on a degenerate orbit

$$\mathcal{O}(\mu_3=0,\mu_8) \simeq \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)\times\mathrm{U}(1)}.$$

Mappings of topological charge 2 can be modeled by the holomorphic functions

$$w_2(z) = \frac{a_1}{z-z_1}, \qquad w_3(z) = \frac{a_2}{z-z_2},$$

where $a_1$, $a_2$, $z_1$, and $z_2$ are fixed complex numbers.

The components $\mu_3$ and $\mu_8$ of the mean field, whose boundary values are called order parameters, are presented in Figs. 2 and 3 (we show a cut along the straight line joining poles of the functions $w_2(z)$ and $w_3(z)$, the origin of coordinates being at a middle of the interval $z_1z_2$). In Figs. 2 and 3, $q$ is a value of the component $\mu_8$ at an initial point of a degenerate orbit ($\mu_3=0$).

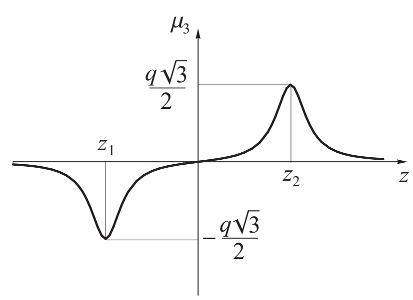


Fig. 2. Contour of $\mu_3(z,\bar{z})$, $\mu_3(\infty)\to 0$

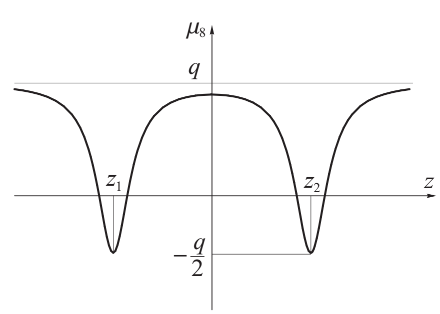

Fig. 3. Contour of $\mu_8(z,\bar{z})$, $\mu_8(\infty)\to q$

These excitations are analogous to Belavin–Polyakov solitons, well-known for planar Heisenberg ferromagnets (the quantities $a_1$ and $a_2$ define widths of solitons, and the quantities $z_1$, $z_2$ give their positions). It is easy to see that an energy of a configuration does not depend on a width of soliton, that proves scale invariance of the energy in the two-dimensional case. Hence, topological perturbations can have arbitrary large size. Such instability (an unrestricted increase of soliton width without pumping of energy) can cause a destruction of the nematic order in the considered model.

## 6. Appendix. Integrability of SU(3)-Symmetric Equations of the Landau–Lifshits Type in a One-Dimensional Space

It is known that system of equations (12) in the one-dimensional case is an integrable Hamiltonian system on a degenerate orbit of the group SU(3) [17]. We generalize Eq. (12) to the case of a generic orbit. The algebraic-geometric nature of equations like (12) is revealed in the frame of so called orbit approach to nonlinear Hamiltonian systems. Below, we construct integrable Hamiltonian equations on orbits of the group SU(3).

Consider polynomials in a complex variable $\lambda$, whose coefficients are anti-Hermitian matrices of the algebra $\mathfrak{su}(3)$. We denote the set of polynomials by $\widetilde{\mathfrak{g}}_+\simeq\mathfrak{su}(3)\otimes\mathcal{P}(\lambda)$, where $\mathcal{P}(\lambda)$ is a ring of polynomials with the standard multiplication operation. If $A,\,B\in\widetilde{\mathfrak{g}}_+$ have the form $A(\lambda)=\sum_n \hat{A}_n\lambda^n$, $B(\lambda)=\sum_k \hat{B}_k\lambda^k$, then

$$[A,\,B]=\sum_{n,k}[\hat{A}_n,\,\hat{B}_k]\lambda^{n+k}\in\widetilde{\mathfrak{g}}_+. \tag{16}$$

Operation (16) shows a structure of graded Lie algebra in $\widetilde{\mathfrak{g}}_+$. Let $X_a^n=\lambda^n\hat{X}_a$ be a basis in $\widetilde{\mathfrak{g}}_+$, where $\hat{X}_a=-\frac{i}{2}\hat{\Lambda}_a$, $a=1,\,2,\,\ldots,\,8$, $\{\hat{\Lambda}_a\}_{a=1}^8$ denote the Gell-Mann matrices.

In $\widetilde{\mathfrak{g}}_+$ we introduce a bilinear ad-invariant form

$$\langle A,B\rangle=-2\,\mathrm{res}\,\lambda^{-N-2}\,\mathrm{Tr}\,A(\lambda)B(\lambda), \tag{17}$$

where $N+1$ is the maximum degree of matrix polynomials $A$ and $B$. Let $\mathcal{M}=(\widetilde{\mathfrak{g}}_+)^*$ be a space dual to $\widetilde{\mathfrak{g}}_+$ with respect to form (17). A collection of the linear forms

$$\xi(\lambda)=\sum_{n=0}^{N}\sum_{a=1}^{8}\xi_a^n\lambda^n\hat{X}_a+\left(\xi_3\hat{X}_3+\xi_8\hat{X}_8\right)\lambda^{N+1}$$

creates a closed ad-invariant subspace $\mathcal{M}^N$ in $\mathcal{M}$. The coordinates $\xi_a^n$ of $\xi(\lambda)\in\mathcal{M}^N$ are calculated by the formula

$$\xi_a^n=\langle\xi(\lambda),X_a^{-n+N+1}\rangle.$$

In the linear space $\mathcal{M}^N$ with coordinates $\xi_a^n$, $n=0,\,1,\ldots,N$, we define the Lie–Poisson bracket

$$\{f_1,f_2\}=\sum_{m,n}\sum_{a,b}^{8}W_{ab}^{mn}\frac{\partial f_1}{\partial\xi_a^m}\frac{\partial f_2}{\partial\xi_b^n} \tag{18}$$

with the Poisson tensor field

$$W_{ab}^{mn}=\langle\xi(\lambda),[X_a^{-m+N+1},X_b^{-n+N+1}]\rangle.$$

We also define two ad-invariant functions $I_2(\lambda)$ and $I_3(\lambda)$ by the formulas

$$I_2(\lambda)=-2\,\mathrm{Tr}\,\xi^2(\lambda)=\sum_a\xi_a^2(\lambda),$$

$$I_3(\lambda)=-4i\,\mathrm{Tr}\,\xi^3(\lambda)=d_{abc}\xi_a(\lambda)\xi_b(\lambda)\xi_c(\lambda),$$

Here, $d_{abc}=-2i\,\mathrm{Tr}(X_aX_bX_c+X_bX_aX_c)$, and $\xi_a(\lambda)$ is a polynomial

$$\xi_a(\lambda)=\xi_a^0+\xi_a^1\lambda+\xi_a^2\lambda^2+\cdots+\xi_a^{N+1}\lambda^{N+1}.$$

The invariant functions are also polynomials in the complex parameter $\lambda$:

$$I_2(\lambda)=h_0+h_1\lambda+\cdots+h_{2N+2}\lambda^{2N+2},$$

$$I_3(\lambda)=f_0+f_1\lambda+\cdots+f_{3N+3}\lambda^{3N+3}.$$

It is easy to prove that the coefficients $h_0,\,\ldots,\,h_{N+1}$, $f_0,\,\ldots\,f_{N+1}$ are annihilators of bracket (18). Fixing them we obtain the system of algebraic equations

$$h_n=\mathrm{const},\;\; f_n=\mathrm{const},\;\; n=0,\ldots,N+1, \tag{19}$$

which give an embedding of the orbit $\mathcal{O}^{N+1}$ of dimension $6(N+1)$ into the linear space $\mathcal{M}^{N+1}$. The rest of coefficients $\{h_{N+2},\,\ldots,\,h_{2N+2},\,f_{N+2},\,\ldots,\,f_{3N+3}\}$ form a pairwise commutative collection of integrals of motion, which is necessary to integrate the Hamiltonian system. We are interested in the functions $h_{N+2}$ and $h_{N+3}$ and the corresponding Hamiltonian equations. In particular, the Hamiltonian $h_{N+2}$ gives rise to so-called stationary equations. In terms of the coordinates $\xi_a^n$, they are

$$\frac{\partial\xi_a^n}{\partial x}=2f_{abc}\xi_c^0\xi_b^{n+1}, \tag{20}$$

where $\{f_{abc}\}$ are antisymmetric structure constants of the algebra $\mathfrak{su}(3)$ in the basis of Gell-Mann matrices:

$$[X_a,X_b]=f_{abc}X_c,$$

$$f_{123} = 1,\ f_{458} = f_{786} = \frac{\sqrt{3}}{2},$$
$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2},$$

By $x$ we denote a "time" with respect to the Hamiltonian $h_{N+2}$, which corresponds to evolution equations

$$\frac{\partial \xi_a^n}{\partial t} = 2 f_{abc}(\xi_c^0 \xi_b^{n+2} + \xi_c^1 \xi_b^{n+1}). \qquad (21)$$

Equations (20) and (21) are consistent, because the corresponding Hamiltonians commute: $\{h_{N+2}, h_{N+3}\} = 0$. Hence, we can determine evolution (21) on trajectories of system (20), i.e. we suppose the dynamical variables $\xi_a^n$ in Eq. (21) to be dependent on $x$. Combining (20) and (21), we have

$$\frac{\partial \xi_a^0}{\partial t} = 2 f_{abc} \xi_c^0 \xi_b^2 = \frac{\partial \xi_a^1}{\partial x}. \qquad (22)$$

The variables $\{\xi_a^1\}$ can be expressed in terms of $\{\xi_a^0\}$ and $\{\frac{\partial}{\partial x}\xi_a^0\}$, then (22) becomes a closed system of partial differential equations for $\{\xi_a^0\}$. For $\{\xi_a^1\}$ it is necessary to solve the degenerate system of equations

$$\frac{\partial \xi_a^0}{\partial x} = 2 f_{abc} \xi_c^0 \xi_b^1, \qquad (23)$$

that becomes possible after restriction to an orbit $\mathcal{O}^{N+1} \subset \mathcal{M}^{N+1}$.

### *6.1. Classification of orbits*

It follows from Eqs. (19) that the orbit $\mathcal{O}^{N+1}$ in $\mathcal{M}^{N+1}$ has the structure of a vector bundle over a co-adjoint orbit of the group SU(3). Hence, a classification of the orbits $\mathcal{O}^{N+1}$ is reduced to that of orbits of SU(3).

Since the group SU(3) is simple, we have $\mathfrak{g}^* = \mathfrak{g}$. Therefore, we consider $\{\xi_a^0\}$ also as coordinates in the algebra $\mathfrak{g} \simeq \mathfrak{su}(3)$. Then a generic element $\hat{\xi} \in \mathfrak{su}(3)$ is represented by the matrix

$$\widehat{\xi} = -\frac{i}{2}\begin{pmatrix} \xi_3^0 + \frac{1}{\sqrt{3}}\xi_8^0 & \xi_1^0 - i\xi_2^0 & \xi_4^0 - i\xi_5^0 \\ \xi_1^0 + i\xi_2^0 & -\xi_3 + \frac{1}{\sqrt{3}}\xi_8 & \xi_6^0 - i\xi_7^0 \\ \xi_4^0 + i\xi_5^0 & \xi_6^0 + i\xi_7^0 & -\frac{2}{\sqrt{3}}\xi_8 \end{pmatrix}. \qquad (24)$$

Let $\hat{\xi}(0)$ be a fixed element of $\mathfrak{su}(3)$. By definition, the set $\mathcal{O}_{\hat{\xi}(0)} = \{g\hat{\xi}(0)g^{-1},\ \forall g \in \mathrm{SU}(3)\}$ is an orbit of SU(3) through the point $\hat{\xi}(0)$. Elements $g'$ of the group SU(3) such that $g'\hat{\xi}(0)g'^{-1} = \hat{\xi}(0)$ form the stationary subgroup at the point $\hat{\xi}(0)$. An orbit $\mathcal{O}_{\hat{\xi}(0)}$, being a homogeneous space, is a coset space $\mathrm{SU}(3)/\mathrm{G}' \simeq \mathcal{O}_{\hat{\xi}(0)}$, where $\mathrm{G}'$ denotes the stationary subgroup.

A maximal commutative subalgebra of a semisimple algebra $\mathfrak{g}$, which is called Cartan, can be diagonalized. The Cartan subalgebra of $\mathfrak{su}(3)$ is formed from diagonal matrices depending on two parameters $\xi_3^0$ and $\xi_8^0$. It is well known that a proper transformation puts any element $\hat{\xi} \in \mathfrak{g}$ into the Cartan subalgebra of $\mathfrak{g}$. This yields that *each orbit intersects the Cartan subalgebra at least once*. In fact, there is more than one intersection point number, more precisely as many as an *order* of the Weyl group W(G). We discuss this in what follows.

Nontrivial similarity transformations $\hat{\xi} \to g\hat{\xi}g^{-1}$ that preserve a subalgebra $\mathfrak{h}$ form a discrete subgroup $\mathrm{W(G)} \subset \mathrm{G}$, which is called a Weyl group [18]. An action of the group W(G) on the subalgebra $\mathfrak{h} = \mathfrak{h}^*$ is generated by reflections in planes orthogonal to simple roots. The Weyl group of SU(3) is generated by two reflections $\sigma_1$ and $\sigma_2$ in the planes shown in Fig. 4 by dotted lines. The full Weyl group consists of six elements $\{e, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1 \simeq \sigma_2\sigma_1\sigma_2\}$, and is isomorphic to the group of permutations $S_3$, $\mathrm{ord}\, S_3 = 3!$.

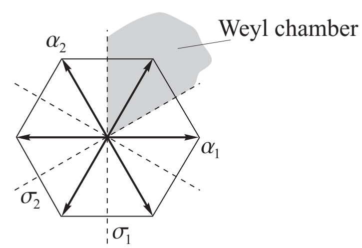


Fig. 4. Root diagram of the group SU(3)

Since a Weyl group acts over a Cartan subalgebra, every point $\sigma\hat{\xi}(0)\sigma^{-1}$, $\sigma \in \mathrm{W(G)}$, belongs to an orbit through $\hat{\xi}(0) \in \mathfrak{h}$. The group W(G) acts efficiently (change an element $\hat{\xi}$ into another, if the element does not belong to reflection planes). An open domain in a Cartan subalgebra where a Weyl group acts efficiently, is called *a Weyl chamber* (see Fig. 4). Elements of different Weyl chambers are adjoint by elements $\sigma \in \mathrm{W(G)}$, this implies that each orbit through a point $\hat{\xi}(0)$ of Weyl chamber intersects a Cartan subalgebra as many times as an order of W(G). If $\hat{\xi}(0)$ is an interior point point of a Weyl chamber, we call the orbit a *generic* one, and call the points $\sigma\hat{\xi}(0)\sigma^{-1}$, $\forall\sigma \in \mathrm{W(G)}$ *poles of an orbit*.

For a generic orbit a stationary subgroup $\mathrm{G}'$ coincides with a maximum torus $\mathrm{T}^r$ for a group G of rank $r$. In the case of group SU(3), we have $\mathrm{T}^2 = \mathrm{U}(1) \times \mathrm{U}(1)$. Hence, generic orbits are coset spaces $\mathcal{O}_{\mathrm{gen}} \simeq \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$.


ISSN 2071-0186. Ukr. J. Phys. 2008. V. 53, N 12

If an initial point $\hat{\xi}(0)$ belongs to a wall of a Weyl chamber (in the case of SU(3), belongs to one of the reflection lines), then we deal with a degenerate orbit. In this case, a stationary subgroup G$'$ contains a semisimple subgroup generated by roots orthogonal to an initial point $\hat{\xi}(0)$. Consider the group SU(3). If $\hat{\xi}(0)$ lies the vertical reflection line, $\alpha_1$ and $-\alpha_1$ are orthogonal to this element. The corresponding $\mathfrak{sl}(2)$-triple $\{X_{\alpha_1}, X_{-\alpha_1}, H_{\alpha_1} = [X_{\alpha_1}, X_{-\alpha_1}]\}$ generates a subgroup $\mathrm{SU}(2) \subset \mathrm{SU}(3)$. Obviously, the element $\hat{\xi}(0) = \frac{-i}{2\sqrt{3}}\,\xi_8^0\,\mathrm{diag}(1,1,-2)$ is invariant under a transformation $g'\hat{\xi}(0)g'^{-1}$, where $g'$ is the unitary matrix

$$g' = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta^* & \alpha^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 & 0 \\ 0 & e^{i\varphi/2} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix}.$$

Hence, $g' \in \mathrm{SU}(2) \times \mathrm{U}(1)$, and a degenerate orbit is a coset space $\mathcal{O}_{\mathrm{deg}} \simeq \mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$.

### *6.2. Equations on orbits and their Hamiltonians*

Return to construction of integrable systems on orbits of loop groups, and consider Eq. (22). In order to solve this equation we have to restrict the degenerate system (23) into an orbit and solve.

If $\xi_a^0 \in \mathcal{O}_{\mathrm{deg}}$, then the matrix $2f_{abc}\xi_c^0$ has rank 4, and its inversion gives a solution

$$\xi_a^1 = \tfrac{2}{3h_0}\, f_{abc}\xi_b^0\xi_{c,x}^0 + \tfrac{h_1}{2h_0}\,\xi_a^0,$$

where the constants $h_0$, $h_1$ define an orbit by Eqs. (19).

If $\xi_a^0 \in \mathcal{O}_{\mathrm{gen}}$, then the matrix $2f_{abc}\xi_c^0$ has rank 6 and is invertable on a generic orbit. Then we have

$$\xi_a^1 = \tfrac{1}{2(h_0^3-3f_0^2)}\Big(h_0^2 f_{abc}\xi_b^0\xi_{c,x}^0 + 3h_0 f_{abc}\eta_b^0\eta_{c,x}^0 - 6f_0 f_{abc}\xi_b^0\eta_{c,x}^0\Big) + \tfrac{2f_0f_1-3h_0^2h_1}{6(f_0^2-h_0^3)}\,\xi_a^0 + \tfrac{3f_0h_1-2h_0f_1}{6(f_0^2-h_0^3)}\,\eta_a^0,$$

where $\eta_a^0 = d_{abc}\xi_b^0\xi_c^0$, and the constants $h_0$, $h_1$, $f_0$, $f_1$ come from Eqs. (19).

Substituting the obtained expressions in the right-hand side of (22), we get two equations for the functions $\xi_a(x,t) \equiv \xi_a^0$:

$$\frac{\partial \xi_a}{\partial t} = \tfrac{2}{3h_0}\, f_{abc}\xi_b\xi_{c,xx} + \tfrac{h_1}{h_0}\,\xi_{a,x}, \quad \xi_a \in \mathcal{O}_{\mathrm{deg}}, \tag{25}$$

$$\frac{\partial \xi_a}{\partial t} = \tfrac{1}{2(h_0^3-3f_0^2)}\Big(h_0^2 f_{abc}\xi_b\xi_{c,xx} - 3f_0 f_{abc}\xi_b\eta_{c,xx} + 3h_0 f_{abc}\eta_b\eta_{c,xx} - 3f_0 f_{abc}\eta_b\xi_{c,xx}\Big) + \tfrac{2f_0f_1-3h_0^2h_1}{6(f_0^2-h_0^3)}\,\xi_{a,x} + \tfrac{3f_0h_1-2h_0f_1}{6(f_0^2-h_0^3)}\,\eta_{a,x}, \quad \xi_a \in \mathcal{O}_{\mathrm{gen}}. \tag{26}$$

Let $h_1 = 0$ in Eq. (25), and replace the variables $\xi_a$ by $\mu_a$. Then its generalization to the two-dimensional case gets the form

$$\frac{\partial \mu_a}{\partial t} = \tfrac{1}{6h_0}\, C_{abc}\mu_b\Delta\mu_c,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Obviously, this equation has the Hamiltonian

$$\mathcal{H}^{\mathrm{eff}} = \tfrac{1}{12h_0}\iint \big(\mu_{a,x}^2 + \mu_{a,y}^2\big)\,dxdy.$$

It is easy to see that (25) coincides with a one-dimensional analog of (12). In other words, Eq. (12) can be considered as a two-dimensional generalization of the integrable equation (25).

In the same way we treat with Eq. (26), namely, replace $\xi_a$ by $\mu_a$ and assign $f_1 = h_1 = 0$. Generalized to two dimensions, the obtained equations get the form

$$\frac{\partial \mu_a}{\partial t} = \tfrac{1}{8(h_0^3-3f_0^2)}\Big(h_0^2 C_{abc}\mu_b\Delta\mu_c - 3f_0 C_{abc}\widetilde{\eta}_b\Delta\mu_c + 3h_0 C_{abc}\widetilde{\eta}_b\Delta\widetilde{\eta}_c - 3f_0 C_{abc}\mu_b\Delta\widetilde{\eta}_c\Big). \tag{27}$$

Here, $\widetilde{\eta}_a$ are quadratic forms in $\mu_a$: $\widetilde{\eta}_a = \widetilde{d}_{abc}\mu_b\mu_c$, where $\widetilde{d}_{abc} = \frac{1}{4}\,\mathrm{Tr}(P_aP_bP_c + P_bP_aP_c)$. Obviously, Eq. (27) is Hamiltonian, and give the following effective Hamiltonian

$$\mathcal{H}^{\mathrm{eff}} = \tfrac{1}{16(h_0^3-3f_0^2)}\iint \big(h_0^2\mu_{a,x}^2 + h_0^2\mu_{a,y}^2 - 6f_0\mu_{a,x}\widetilde{\eta}_{a,x} - 6f_0\mu_{a,y}\widetilde{\eta}_{a,y} + 3h_0\widetilde{\eta}_{a,x}^2 + 3h_0\widetilde{\eta}_{a,y}^2\big)\,dxdy.$$

## 7. Conclusions

In the present paper, we have constructed nonlinear stationary excitations appearing in the nematic phase of a planar magnet of spin $s = 1$, modeled by a square lattice with a biquadratic interaction between nearest-neighbor sites. These excitations are characterized by an integer topological charge, and reveal themselves as regions with nonzero magnetization and mean quadrupole moment. Topological excitations in a two-dimensional system (without taking into account an anisotropy and a demagnetizing field) can increase unrestrictedly without pumping of energy. This destroys

a nematic state in the system, according to the Mermin–Wagner theorem on absence of a long-range order in one- and two-dimensional systems.

This work is partly supported by grant DFFD UkrF16/457-2007 and the grant of the International Charitable Fund for Renaissance of Kyiv-Mohyla Academy.

1. L. Onsager, Phys. Rev. **65**, 117 (1944).
2. J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).
3. V.L. Berezinskii, Zh. Eksp. Teor. Fiz. **59**, 907 (1971); Ibid. **61**, 1144 (1971).
4. A.M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge Univ. Press, Cambridge, 1995).
5. M. Blume and Y.Y. Hsien, J. Appl. Phys. **40**, 1249 (1969).
6. H.H. Chen and P.M. Levy, Phys. Rev. B **7**, 4267 (1974).
7. V.M. Matveev, Zh. Eksp. Teor. Fiz. **65**, 1626 (1973).
8. M. Nauciel-Bloch, G. Sarma, and A. Castets, Phys. Rev. B **5**, 4603 (1972).
9. E.A. Harris and J. Owen, Phys. Rev. Lett. **11**, 9 (1963).
10. D.S. Rodbell, I.S. Jacobs, and J. Owen, Phys. Rev. Lett. **11**, 10 (1963).
11. N.N. Bogolyubov and N.N. Bogolyubov (jr.), *Introduction in Quantum Statistical Mechanics* (Nauka, Moscow, 1984) (in Russian).
12. K. Buchta, G. Fath, and O. Hegezand, Phys. Rev. B **72**, 054433 (2005).
13. V.M. Loktev and V.S. Ostrovsky, Fiz. Nizk. Temp. **20**, 983 (1994).
14. C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).
15. V.G. Bar'yakhtar, V.N. Krivoruchko, and D.A. Yablonsky, *Green Functions in Magnetism Theory* (Naukova Dumka, Kiev, 1984) (in Russian).
16. L.D. Landau and E.M. Lifshits, *Electrodynamics of Continuous Media* (New York: Pergamon, 1984).
17. P.I. Holod and O. Kisilevych, *Integrable Dynamics of SU*(3) *Magnets*, Preprint ITF-93-30U (Inst. Theor. Phys., Kiev, 1993) (in Ukrainian).
18. S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press, New York, 1962).

Received 02.04.08.
Translated from Ukrainian by V.V. Kukhtin

## ВПОРЯДКОВАНІ СТАНИ ТА НЕЛІНІЙНІ ВЕЛИКОМАСШТАБНІ ЗБУДЖЕННЯ У ПЛОСКОМУ МАГНЕТИКУ ЗІ СПІНОМ $s=1$

*Ю.М. Бернацька, П.І. Голод*

Р е з ю м е

Досліджено впорядковані стани та топологічні збудження у квазідвовимірному магнетику, змодельованому квадратною ґраткою зі спінами $s=1$ у вузлах та гамільтоніаном з біквадратною обмінною взаємодією найближчих сусідів. Запропоновано два ефективних гамільтоніани для опису великомасштабних збуджень у строго двовимірному випадку. Один з них описує збудження середнього поля в нематичній фазі, інший — у змішаній феромагнітно-нематичній фазі. Показано, що ефективні гамільтоніани мінімізуються на конфігураціях, які мають фіксований топологічний заряд. Ці топологічні збудження можуть виникати при незначних температурах і бути причиною руйнування дальнього порядку у строго двовимірній системі.